Homework 4
(50 points)

1. **(15 + 5 points) Most Functions are Small Biased.** Let $f\colon \{0,1\}^n \to \{+1,-1\}$ be a boolean function. Our objective is to consider a random boolean function f. A random boolean function (the distribution is represented by $\mathbb{F}_n$) is generated as follows.

   For every input $x \in \{0,1\}^n$, choose the value of $f(x)$ independently to be $+1$ with probability $1/2$, and -1 with probability $1/2$.

   (a) Fix $S \in \{0,1\}^n$. For every $x \in \{0,1\}^n$, note that $f(x) \cdot \chi_S(x)$ is independently $+1$ with probability $1/2$, and -1 with probability $1/2$. Using this observation, upper-bound the following probability
   $$\mathbb{P}\left[\widehat{f}(S) \geqslant \varepsilon \colon f \sim \mathbb{F}_n\right]$$

   (b) Use the previous result to lower-bound the following probability
   $$\mathbb{P}\left[\forall S \in \{0,1\}^n, \widehat{f}(S) \leqslant \varepsilon \colon f \sim \mathbb{F}_n\right]$$

2. **(10 + 20 points) Monotonicity of Norm.** Define the following function

$$f(p) = \frac{1}{p} \log \left(\frac{1}{n} \sum_{i=1}^{n} a_i^p \right)$$

Note that in this function we have fixed the values of $a_1, \dots, a_n$.

(a) Calculate $\frac{\mathrm{d}f}{\mathrm{d}p}$.

(b) Use Jensen's inequality to prove that $\frac{\mathrm{d}f}{\mathrm{d}p} \geqslant 0$, for $p \geqslant 1$. And equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

This proves that the norm is non-decreasing, and equality holds if and only if all a_is are equal!